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On "new" turning points associated with regular singular points in the exact WKB analysis

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1 Introduction

In this report we shall consider the following equation near the origin:

$$\left(-\frac{d^2}{dx^2}+\eta^2\left(Q_0(x)+\eta^{-1}\frac{Q_1(x)}{x}+\eta^{-2}\frac{Q_2(x)}{x^2}\right)\right)\psi(x)=0, \tag{1.1}$$

where η denotes a large parameter and $Q_j(x)$ $(j=0,1,2)$ denotes a holomorphic function which does not vanish at the origin. Our eventual purpose is to determine the connection formulas of the Borel sum of WKB solutions of (1.1) from a view point of the exact WKB analysis ([V], [S], [DDP], [AKT]); we treat WKB solutions in all orders by giving them an analytic meaning through Borel resummation.

As we shall see below we will find the origin has a structure of a turning point of (1.1). Our argument employed in this report is based on the transformation theory developed in [AKT]; we first discuss the transformation of (1.1) into a canonical equation in §1. Then we give the connection formula for this canonical equation in §2. In §3 we consider the connection formulas for (1.1).

2 Reduction to the canonical equation

In this section we discuss the transformation of (1.1) into the Whittaker-type equation

$$\left(-\frac{d^2}{dx^2}+\eta^2\left(\frac{1}{4}+\eta^{-1}\frac{b}{x}+\eta^{-2}\frac{c}{x^2}\right)\right)\psi=0 \tag{2.1}$$

near the origin. Here $b=\operatorname*{Res}_{x=0}\widetilde{S}_{\text{odd}}(\widetilde{x},\eta)$ and $c=Q_2(0)$, where

$$\widetilde{S}_{\text{odd}}=\eta\sqrt{Q_0(x)}+\frac{Q_2(\widetilde{x})}{2\widetilde{x}\sqrt{Q_0}}+\cdots \tag{2.2}$$

is the odd degree part of the solutions of the Riccati equation

$$\widetilde{S}^2+\frac{d\widetilde{S}}{d\widetilde{x}}=\eta^2\left(Q_0(\widetilde{x})+\eta^{-1}\frac{Q_1(\widetilde{x}}{\widetilde{x}}+\eta^{-2}\frac{Q_2(\widetilde{x})}{\widetilde{x}^2}\right) \tag{2.3}$$

associated with (1.1). Our main result in this section is (cf. [AKT], [K])

Proposition 2.1 *We can find a neighborhood U of the origin and a pre-Borel summable series $x(\widetilde{x},\eta)=x_0(\widetilde{x})+\eta^{-1}x_1(\widetilde{x})+\eta^{-2}x_2(\widetilde{x})+\cdots$ such that each $x_j(\widetilde{x})$ is holomorphic in U and satisfies*

(i) $x_0(0)=0$, $(dx_0/d\widetilde{x})(0)\neq 0$ holds;

(ii) Every $x_j(\widetilde{x})$ vanishes at the origin;

(iii) The following relation formally holds in U;

$$\begin{aligned}Q_0(\widetilde{x})+\eta^{-1}\frac{Q_1(\widetilde{x})}{\widetilde{x}}+\eta^{-2}\frac{Q_2(\widetilde{x})}{\widetilde{x}^2} &= \left(\frac{\partial x(\widetilde{x},\eta)}{\partial\widetilde{x}}\right)^2\left(\frac{1}{4}+\eta^{-1}\frac{b}{x(\widetilde{x},\eta)}+\eta^{-2}\frac{c}{x(\widetilde{x},\eta)^2}\right)\\ &\quad -\frac{1}{2}\eta^{-2}\left\{x(\widetilde{x},\eta);\eta\right\}.\end{aligned} \tag{2.4}$$

Here $b=\operatorname*{Res}_{x=0}\widetilde{S}_{\text{odd}}(\widetilde{x},\eta)$, $c=Q_2(0)$ and $\{x(\widetilde{x},\eta);\eta\}$ denotes the Schwarzian derivative, i.e.

$$\{x(\widetilde{x},\eta);\widetilde{x}\}=\frac{x'''}{x'}-\frac{3}{2}\left(\frac{x''}{x'}\right)^2. \tag{2.5}$$

To prove this proposition we first assume c to be an infinite series of η: $c = c_0 + \eta^{-1}c_1 + \eta^{-2}c_2 + \cdots$. Then by substituting $x(\tilde{x}, \eta)$, b and c into (2.4) and comparing both sides degree by degree, we obtain

$$\frac{1}{4}\left(\frac{dx_0}{d\tilde{x}}\right)^2 = Q_0(\tilde{x}) \tag{2.6.0}$$

for the 0-th degree and

$$2\frac{dx_0}{d\tilde{x}}\frac{dx_n}{d\tilde{x}} = F_n(\tilde{x}) - \frac{1}{x_0}\left(\frac{dx_0}{d\tilde{x}}\right)^2 b_{n-1} - \left(\frac{1}{x_0}\frac{dx_0}{d\tilde{x}}\right)^2 c_{n-2} \tag{2.6.n}$$

for the n-th degree, where $n = 1, 2, 3\cdots$ and we set $c_{-1} = 0$ for convenience. Here

$$\begin{aligned} F_1(\tilde{x}) &= \frac{Q_1(\tilde{x})}{\tilde{x}}, \\ F_2(\tilde{x}) &= \frac{Q_2(\tilde{x})}{\tilde{x}^2} - \frac{1}{4}(x_1')^2 - b_0\frac{2x_0'x_1'x_0 - (x_0')^2 x_0}{(x_0)^2} + \frac{1}{2}\{x_0(\tilde{x}); \eta\}, \end{aligned}$$

and

$$\begin{aligned} F_n(\tilde{x}) = &-\frac{1}{4}\sum_{\substack{\nu_1+\nu_2=n \\ \nu_j \geq 0}} x'_{\nu_1}x'_{\nu_2} \\ &- \sum_{\substack{\mu+\nu+k+l=n-1 \\ \mu,\nu,k,l\geq 0, k\neq n-1}} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu \\ \mu_j\geq 0}} \sum_{\substack{\nu_1+\nu_2=\nu \\ \nu_j\geq 0}} (-1)^l b_k x'_{\nu_1}x'_{\nu_2}\frac{x_{\mu_1+1}\cdots x_{\mu_l+1}}{(x_0)^{l+1}} \\ &- \sum_{\substack{\mu+\nu+k+l=n-2 \\ \mu,\nu,k,l\geq 0, k\neq n-2}} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu \\ \mu_j\geq 0}} \sum_{\substack{\nu_1+\nu_2=\nu \\ \nu_j\geq 0}} (-1)^l(l+1) c_k x'_{\nu_1}x'_{\nu_2}\frac{x_{\mu_1+1}\cdots x_{\mu_l+1}}{(x_0)^{l+2}} \\ &+\frac{1}{2}\sum_{\substack{\mu+k+l=n-2 \\ \mu,k,l\geq 0}} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu \\ \mu_j\geq 0}} (-1)^l x'''_k\frac{x'_{\mu_1+1}\cdots x'_{\mu_l+1}}{(x'_0)^{l+1}} \\ &-\frac{3}{4}\sum_{\substack{\mu+\nu+l=n-2 \\ \mu,\nu,l\geq 0}} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu \\ \mu_j\geq 0}} \sum_{\substack{\nu_1+\nu_2=\nu \\ \nu_1,\nu_2\geq 0}} (-1)^l(l+1) x''_{\nu_1}x''_{\nu_2}\frac{x'_{\mu_1+1}\cdots x'_{\mu_l+1}}{(x'_0)^{l+1}} \end{aligned}$$

The holomorphic solution of (2.6.0) is

$$x_0(\tilde{x}) = 2\int_0^{\tilde{x}} \sqrt{Q_0(\tilde{x})}d\tilde{x}, \tag{2.7}$$

which satisfies the condition (i). Let U be a neighborhood of the origin so that $x_0(\widetilde{x})$, $Q_1(\widetilde{x})$ and $Q_2(\widetilde{x})$ are holomorphic in U.

Next we determine $x_1(\widetilde{x})$. To make a solution of (2.6.1) to be holomorphic near $\widetilde{x} = 0$, we set

$$b_0 = \frac{x_0(\widetilde{x})}{(x_1'(\widetilde{x}))^2} F_1(\widetilde{x})\Big|_{\widetilde{x}=0} \quad \left(= \frac{Q_1(0)}{2\sqrt{Q_0(0)}} \right). \tag{2.8}$$

(Note that $F_1(x)$ has a simple pole at the origin.) Then

$$x_1(\widetilde{x}) = \int_0^{\widetilde{x}} \frac{1}{4\sqrt{Q_0(\widetilde{x})}} \left(F_1(\widetilde{x}) - \frac{(x_0')^2}{x_0} b_0 \right) d\widetilde{x} \tag{2.9}$$

is a holomorphic solution of (2.6.1) in U. Here we have chosen the origin as an end-point of the integration in (2.9); otherwise $F_3(\widetilde{x})$ has a pole at the origin whose degree is greater than two. In this case (2.6.3) admits no holomorphic solutions.

By the same reason we choose c_0 and b_1 so that

$$F_2(\widetilde{x}) - \left(\frac{x_0'}{x_0}\right)^2 c_0 - \frac{(x_0')^2}{x_0} b_1 \tag{2.10}$$

is holomorphic in U. (Note that $F_2(x)$ has a double pole at the origin.) Then

$$x_2(\widetilde{x}) = \int_0^{\widetilde{x}} \frac{1}{4\sqrt{Q_0(\widetilde{x})}} \left(F_2(\widetilde{x})) - \left(\frac{x_0'}{x_0}\right)^2 c_0 - \frac{(x_0')^2}{x_0} b_1 \right) d\widetilde{x} \tag{2.11}$$

is holomorphic in U, and gives the solution of (2.6.2) in U which vanishes at the origin.

In a similar way, we can recursively determine $x_n(\widetilde{x})$ for $n = 3, 4, 5, \cdots$. Since $x_j(\widetilde{x})$ vanishes at the origin for $j = 0, 1, 2, \cdots, n-1$, $F_n(\widetilde{x})$ has a pole of degree, at most, two at the origin. Hence by choosing b_{n-1}, and c_{n-2} appropriately

$$F_n(\widetilde{x}) - \left(\frac{x_0'}{x_0}\right)^2 c_{n-2} - \frac{(x_0')^2}{x_0} b_{n-1} \tag{2.12}$$

becomes holomorphic in U. Then

$$x_n(\widetilde{x}) = \int_0^{\widetilde{x}} \frac{1}{4\sqrt{Q_0(\widetilde{x})}} \left(F_n(\widetilde{x}) - \left(\frac{x_0'}{x_0}\right)^2 c_{n-2} - \frac{(x_0')^2}{x_0} b_{n-1} \right) d\widetilde{x} \tag{2.13}$$

is holomorphic in U, and gives the solution of (2.6.n) which vanishes at the origin.

Thus we have determined $\{x_n(\tilde{x})\}$, $\{b_n\}$ and $\{c_n\}$. Furthermore we can prove $c_j = 0$ for $j = 1, 2, 3, \cdots$. By multiplying both sides of (2.4) by $\tilde{x}^2$ and taking the limit of $\tilde{x}$ tending to 0. Then we obtain

$$\eta^{-2} Q_2(0) = \lim_{\tilde{x}\to 0} \left(\frac{\partial x(\tilde{x}, eta)}{\partial \tilde{x}} \right)^2 \left(\eta^{-2} \frac{\tilde{x}^2}{x(\tilde{x},\eta)^2} c \right) = \eta^{-2} c, \tag{2.14}$$

where we have used $x_j(0) = 0$ for any j.

Let

$$\begin{aligned} S_{\text{odd}} &= \frac{1}{2}\eta + \frac{b_0}{x} + \eta^{-1}\left(\frac{b_1}{x} + \frac{c-(b_0)^2}{x^2} \right) \\ &\quad + \eta^{-2}\left(\frac{b_2}{x} - \frac{2b_0 b_2}{x^2} + \frac{2b_0((b_0)^2 - c + 1)}{x^3} \right) + \cdots \qquad (2.15) \\ \Big(&= \frac{1}{2}\eta + \frac{b}{x} + \eta^{-1}\frac{c-b^2}{x^2} + \eta^{-2}\frac{2b(b^2-c+1)}{x^3} + \cdots \Big) \qquad (2.16) \end{aligned}$$

be the odd degree part of solutions of the Riccati equation associated with (2.1). Then we can prove that the following formally holds (cf. [KT]):

$$\tilde{S}_{\text{odd}}(\tilde{x}, \eta) = \frac{\partial x(\tilde{x}, \eta)}{\partial \tilde{x}} S_{\text{odd}}(x(\tilde{x}, \eta), \eta). \tag{2.17}$$

As a corollary of (2.17) we obtain

$$\operatorname*{Res}_{\tilde{x}=0} S_{\text{odd}}(\tilde{x}, \eta) = \operatorname*{Res}_{x=0} S_{\text{odd}}(x, \eta) \ \ (= b). \tag{2.18}$$

The remaining part of the proof is to show the pre-Borel summability of $x(\tilde{x}, \eta)$, which follows in a similar way as in [AKT]. (See also [K]). □

We should note here the relation between WKB solutions of (1.1) and (2.1); as a corollary of (2.17), we can find $C_\pm = C_{\pm,0} + C_{\pm,1}\eta^{-1} + C_{\pm,2}\eta^{-2} + \cdots$ so that the following relation formally holds;

$$\tilde{\psi}_\pm(\tilde{x}, \eta) = C_\pm \left(\frac{\partial x(\tilde{x}, \eta)}{\partial \tilde{x}} \right)^{-1/2} \psi_\pm(x(\tilde{x}, \eta), \eta), \tag{2.19}$$

where

$$\tilde{\psi}_\pm(\tilde{x}, \eta) = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\left(\pm\eta \int_0^{\tilde{x}} \sqrt{Q_0(\tilde{x})} d\tilde{x} \right) \exp\left(\pm \int_0^{\tilde{x}} \left(S_{\text{odd}} - \eta\sqrt{Q_0(\tilde{x})} \right) d\tilde{x} \right) \tag{2.20}$$

are WKB solutions of (1.1) and

$$\psi_\pm(x,\eta) = \frac{1}{\sqrt{S_{\text{odd}}}} x^{\pm b} \exp\left(\pm\frac{1}{2}\eta\right) \exp\left(\pm\int_\infty^x \left(S_{\text{odd}} - \frac{1}{2}\eta x - \frac{b}{x}\right) dx\right) \tag{2.21}$$

are WKB solutions of (2.1). Here x_0 is an appropriate reference point. (Hence $C_\pm$ depend on x_0.)

Keeping these relations (2.19) in mind, we shall consider the connection problem of WKB solutions of the canonical equation in the next section.

3 Connection formulas for the canonical equation

Throughout this section we assume b is a complex number (i.e., independent of η), and we shall consider the following WKB solutions of (2.1):

$$\psi_\pm(x,\eta) = \frac{1}{\sqrt{S_{\text{odd}}}} x^{\pm b} e^{\pm\eta x/2} \exp\left(\pm\int_\infty^x \left(S_{\text{odd}} - \frac{1}{2}\eta - \frac{b}{x}\right)\right) \tag{3.1}$$

where

$$S_{\text{odd}} = \frac{1}{2}\eta + \frac{b}{x} + \eta^{-1}\frac{c-b^2}{x^2} + \eta^{-2}\frac{2b(b^2-c+1)}{x^3} + \cdots. \tag{3.2}$$

We choose the principal branch for $x^{\pm b}$, i.e., $x^{\pm b}$ is positive along the positive real axis.

We define a Stokes curve of (2.1) emanating from the origin by $\Im \int_0^x \sqrt{\frac{1}{2}} dx = 0$ (hence $\Im x = 0$). By its definition two Stokes curves emanate form the origin; one is the positive real axis, and another is the negative real axis.

Proposition 3.1 *WKB solutions $\psi_\pm$ are Borel summable except for the positive axis. Let $\psi_\pm^I$ denote the Borel-summed WKB solutions in the lower half plane, $\psi_\pm^{II}$ the Borel-resuumed WKB solutions in the upper half plane. Then the analytic continuation of ψ_+^I (resp. ψ_-^I) across the positive real axis becomes*

$$\psi_+^{II} + \frac{2i\pi}{\Gamma(\kappa+\mu+1/2)\Gamma(\kappa-\mu+1/2)}\eta^{2\kappa}\psi_-^{II} \tag{3.3}$$

(resp. ψ_-^{II}), where $\kappa = -b$ and $\mu = \sqrt{c+1/2}$. The analytic continuation of ψ_-^{II} (resp. ψ_+^{II}) across the negative real axis is

$$\psi_+^I + \frac{2i\pi e^{-2i\kappa}}{\Gamma(-\kappa+\mu+1/2)\Gamma(-\kappa-\mu+1/2)}\eta^{-2\kappa}\psi_-^{II} \tag{3.4}$$

(resp. ψ_+^I).

Proof For the calculational convenience we consicder WKB solutions of (2.1) normalized as $\varphi_\pm = \eta^{\pm b}\psi_\pm$. Then $\varphi_\pm$ have an expansion of the form

$$\varphi_\pm = \sqrt{2}\,(\eta x)^{\pm b}\, e^{\pm\frac{1}{2}x}\sum_{j=0}^{\infty}\varphi_{\pm,j}x^{-j}\eta^{-j-1/2}, \tag{3.5}$$

where $\varphi_{\pm,j}$ are constants and $\varphi_{\pm,0} = 1$. Their Borel transforms $\varphi_{\pm,B}$ becomes

$$\varphi_{\pm,B}(x,y) = \sqrt{\frac{2}{x}}\sum_{j=0}^{\infty}\frac{\varphi_{\pm,j}}{\Gamma(j\mp\kappa-\frac{1}{2})}\left(\frac{y}{x}\pm\frac{1}{2}\right)^{j\mp\kappa-frac12}, \tag{3.6}$$

where $\kappa = -b$. Thus $(2/x)^{-1/2}\varphi_{\pm,B}$ are funcions of y/x, which we denote by $h_\pm(y/x)$ respectively. Since $\varphi_{\pm,B}(x,y)$ satisfy

$$\left(-\frac{\partial^2}{\partial y^2}+\frac{1}{4}\frac{\partial^2}{\partial x^2}-\frac{\kappa}{x}\frac{\partial}{\partial y}+c\right)\varphi_{\pm,B}(x,y) = 0, \tag{3.7}$$

we can verify that $h_\pm(t)$ are solutions of

$$\left(\left(\frac{1}{4}-t^2\right)\frac{d^2}{dt^2}-(\kappa+3t)\frac{d}{dt}+c-\frac{3}{4}\right)h = 0, \tag{3.8}$$

or,

$$\left(s(1-s)\frac{d^2}{ds^2}-\left(\kappa+\frac{3}{2}+3s\right)\frac{d}{ds}+c-\frac{3}{4}\right)h = 0, \tag{3.9}$$

where $s = t+1/2$. By noting (3.5) we conclude

$$\varphi_{+,B}(x,y) = \frac{1}{\Gamma(\kappa+1/2)}\sqrt{\frac{2}{x}}s^{\kappa-1/2}F(\kappa+\mu+\frac{1}{2},\kappa-\mu+\frac{1}{2},\kappa+\frac{1}{2};s)\Big|_{s=\frac{y}{x}+\frac{1}{2}} \tag{3.10}$$

$$\varphi_{-,B}(x,y) = \frac{1}{\Gamma(-\kappa+1/2)}\sqrt{\frac{2}{x}}(s-1)^{-\kappa-1/2} \times F(-\kappa+\mu+\frac{1}{2},-\kappa-\mu+\frac{1}{2},-\kappa+\frac{1}{2};1-s)\Big|_{s=\frac{y}{x}+\frac{1}{2}} \tag{3.11}$$

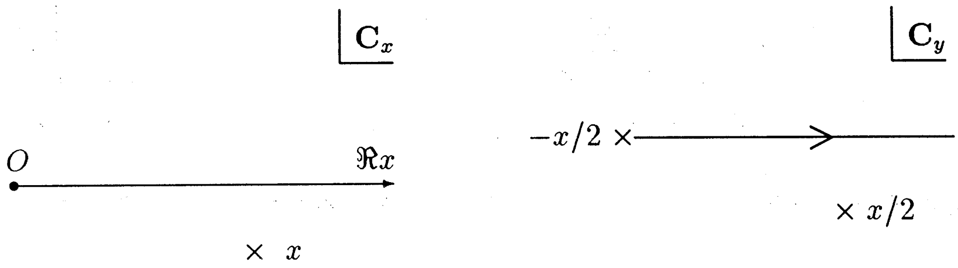

Figure 1:

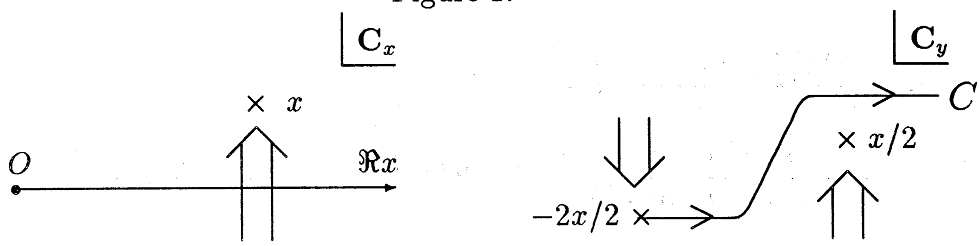

Figure 2:

where $F(\alpha, \beta, \gamma; z)$ designates the Gauss hypergeometric functions and $\mu = \sqrt{c + 1/4}$.

From this explicit descripion of the Borel transforms of WKB solutions, we find $\varphi_{\pm,B}(x, y)$ is holomorphic except for $y = x/2$ and $y = -x/2$, and their Borel sum

$$\varphi_{\pm}(x, \eta) = \int_{\mp x/2}^{\infty} e^{-\eta y} \varphi_{\pm,B}(x, y) dy \tag{3.12}$$

are well-defined except for $\Im x = 0$, i.e., except for the Stokes curves.

We shall now determine the connection formula when we cross the positive real axis. Let x be below the positive real axis as shown in the left of Fig.1. Then the configuration of singularities of $\varphi_{+,B}(x, y)$ and the integration path for the Borel sum of φ_+ is as shown in the right of Fig.1. After we cross the positive real axis, such a configulation changes as shon in Fig.2 and Fig.3.

To determine the singular part of $\varphi_{+,B}(x, y)$ at $y = x/2$, we employ the connecion formula of Gauss Hypergeometric funcions:

$$s^{-1/2} F\left(\alpha - \frac{1}{2}, \beta - \frac{1}{2}, \frac{1}{2}; s\right)$$

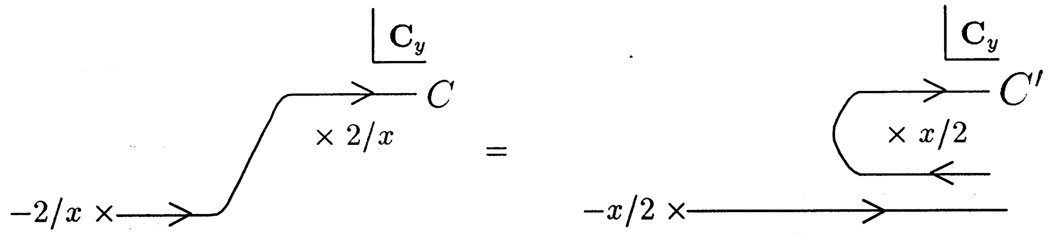

Figure 3:

$$\begin{aligned} &= \frac{\Gamma(\frac{1}{2})\Gamma(-\frac{1}{2})}{\Gamma(1-\alpha)\Gamma(1-\beta)} F\left(\alpha, \beta, \frac{3}{2}; 1-s\right) \\ &\quad + \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\alpha-\frac{1}{2}\right)\Gamma\left(\beta-\frac{1}{2}\right)} (1-s)^{-1/2} F\left(\frac{3}{2}-\alpha, \frac{3}{2}-\beta, \frac{1}{2}; 1-s\right). \end{aligned} \tag{3.13}$$

From this relation, we find that the singular part of $\varphi_{+,B}(x,y)$ at $y = x/2$ is given by

$$\begin{aligned} & \frac{1}{\sqrt{x}} \frac{\sqrt{2}}{\Gamma(\kappa+\frac{1}{2})} \frac{\Gamma(\kappa+\frac{1}{2})\Gamma(\kappa+\frac{1}{2}))}{\Gamma(\kappa+\mu+\frac{1}{2})\Gamma(\kappa-\mu+\frac{1}{2})} \\ & \qquad \left[(1-s)^{-\kappa-1/2} F(-\kappa-\mu+\frac{1}{2}, -\kappa+\mu+\frac{1}{2}, -\kappa+\frac{1}{2}; 1-s)\right]_{s=\frac{y}{x}+\frac{1}{2}} \end{aligned} \tag{3.14}$$

$$\begin{aligned} &= \sqrt{\frac{2}{x}} \frac{\Gamma(\kappa+\frac{1}{2})}{\Gamma(\kappa+\mu+\frac{1}{2})\Gamma(\kappa-\mu+\frac{1}{2})} \\ & \qquad \left[(1-s)^{-\kappa-1/2} F(-\kappa-\mu+\frac{1}{2}, -\kappa+\mu+\frac{1}{2}, -\kappa+\frac{1}{2}; 1-s)\right]_{s=\frac{y}{x}+\frac{1}{2}} \end{aligned} \tag{3.15}$$

Hense the discontinuity $\Delta_{y=x/2}\varphi_{+,B}(x,y)$ of $\varphi_{+,B}(x,y)$ at $y = x/2$ along the cut $\{y \in \mathbf{C}; \Im y = \Im(x/2), \Re y \geq \Re(x/2)\}$ becomes

$$\begin{aligned} \Delta_{y=x/2}\varphi_{+,B}(x,y) &= \sqrt{\frac{2}{x}} \frac{\Gamma(\kappa+\frac{1}{2})}{\Gamma(\kappa+\mu+\frac{1}{2})\Gamma(\kappa-\mu+\frac{1}{2})} 2i\cos(\pi\kappa) \\ & \qquad \left[(s-1)^{-\kappa-1/2} F(-\kappa-\mu+\frac{1}{2}, -\kappa+\mu+\frac{1}{2}, -\kappa+\frac{1}{2}; 1-s)\right]_{s=\frac{y}{x}+\frac{1}{2}} \end{aligned} \tag{3.16}$$

$$= 2i \frac{\Gamma(\kappa+\frac{1}{2})\Gamma(-\kappa+\frac{1}{2})}{\Gamma(\kappa+\mu+\frac{1}{2})\Gamma(\kappa-\mu+\frac{1}{2})} \cos(\pi\kappa)\varphi_{-,B}(x,y) \tag{3.17}$$

$$= \frac{2i\pi}{\Gamma(\kappa+\mu+\frac{1}{2})\Gamma(\kappa-\mu+\frac{1}{2})}\varphi_{-,B}(x,y). \tag{3.18}$$

Thus we obtain the connection formula for $\varphi_{+,B}(x,y)$ when we cross the positive real axis. In a similar way we can determine connection formulars when we cross the negative real axis.

4 Connection formulas for the genral case

In the above secionts we have constructed the pre-Borel summable series which transforms (1.1) to (2.1), and clarifined the behavior of Borel resuumed WKB soluitons of the canonical equation. Following the definition for the canonical equation, we define the Stokes curves for (1.1) emanating from the origin by

$$\Im \int_0^x \sqrt{Q_0(x)}dx = 0. \tag{4.1}$$

Then two Stokes curves emanates from the origin.

Let $\widetilde{\psi}_\pm$ be WKB solutions (2.20) of (1.1), and γ a Stokes curve emanating from the origin. Having the result obtained so far, it may be expected that when the Borel sum of $\widetilde{\psi}_\pm$ crosses γ in a couterclockwise manner with respect to the center $\widetilde{x}=0$, we obtain

$$\widetilde{\psi}_+ \mapsto \widetilde{\psi}_+ + \frac{2i\pi}{\Gamma(\kappa+\mu+1/2)\Gamma(\kappa-\mu+1/2)}\frac{C_+}{C_-}\eta^{2\kappa}\widetilde{\psi}_-, \tag{4.2}$$

$$\widetilde{\psi}_- \mapsto \widetilde{\psi}_-, \tag{4.3}$$

$$\tag{4.4}$$

if $\Re\int_0^{\widetilde{x}}\sqrt{Q_0(\widetilde{x})}d\widetilde{x}$ is positive along the γ, and

$$\widetilde{\psi}_+ \mapsto \widetilde{\psi}_+, \tag{4.5}$$

$$\widetilde{\psi}_- \mapsto \widetilde{\psi}_+ + \frac{2i\pi e^{-2i\kappa}}{\Gamma(-\kappa+\mu+1/2)\Gamma(-\kappa-\mu+1/2)}\frac{C_-}{C_+}\eta^{-2\kappa}\widetilde{\psi}_-, \tag{4.6}$$

$$\tag{4.7}$$

if $\Re\int_0^{\widetilde{x}}\sqrt{Q_0(\widetilde{x})}d\widetilde{x}$ is negative along the γ.

To give the proof of these formulas, we must give the analytic meaning to (2.19); by Taylor expansion (2.19) becomes

$$\widetilde{\psi}_{\pm,B}(\widetilde{x},y) = A(\widetilde{x};\frac{\partial}{\partial\widetilde{x}},\frac{\partial}{\partial y})\psi_{\pm,B}(x_0(\widetilde{x}),y), \tag{4.8}$$

in the Borel plane. Here $A(\tilde{x};\partial/\partial\tilde{x},\partial/\partial y)$ is a microdifferetial operator. The problerm we have not confirmed is that the domain of this microdifferential operator $A(\tilde{x};\partial/\partial\tilde{x},\partial/\partial y)$ is so large that the relation (2.19) becomes an analytic one. In fact, if we can show this claim, the following holds: for a sufficiently small neighborhood W of the origin of $\mathbf{C}_{\tilde{x}}\times\mathbf{C}_y$, both $\tilde{\psi}_{+,B}(\tilde{x},y)$ and $\tilde{\psi}_{-,B}(\tilde{x},y)$ have their singularities in W only along $\{(\tilde{x},y)\in W; y=\pm\int_0^{\tilde{x}}\sqrt{Q_0(\tilde{x})}d\tilde{x}\}$. Furthermore the discontinuity of $\tilde{\psi}_{+,B}(\tilde{x},y)$ (resp. $\tilde{\psi}_{-,B}(\tilde{x},y)$) along the cut $\{(\tilde{x},y)\in W;\Im y=\Im(\int_0^{\tilde{x}}\sqrt{Q_0(\tilde{x})}d\tilde{x}),\Re y\geq\Re(\int_0^{\tilde{x}}\sqrt{Q_0(\tilde{x})}d\tilde{x})\}$ (resp. $\{(\tilde{x},y)\in W;\Im y=\Im(-\int_0^{\tilde{x}}\sqrt{Q_0(\tilde{x})}d\tilde{x}),\Re y\geq\Re(-\int_0^{\tilde{x}}\sqrt{Q_0(\tilde{x})}d\tilde{x})\}$) coincides with

$$\frac{2i\pi}{\Gamma(\kappa+\mu+1/2)\Gamma(\kappa-\mu+1/2)}\frac{C_+}{C_-}\eta^{2\kappa}\tilde{\psi}_{B,-}(\tilde{x},y) \tag{4.9}$$

(resp.

$$\frac{2i\pi e^{-2i\kappa}}{\Gamma(-\kappa+\mu+1/2)\Gamma(-\kappa-\mu+1/2)}\frac{C_-}{C_+}\eta^{-2\kappa}\tilde{\psi}_{-,B}(\tilde{x},y)) \tag{4.10}$$

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